POLYHEDRAL BANACH SPACES AND EXTENSIONS OF COMPACT OPERATORS

BY

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ABSTRACT

Let X be a polyhedral Banach space whose dual is an $L_1(\mu)$ space for some measure μ. Then for each Banach spaces $Y \subseteq Z$ and each compact operator T: $Y \to X$ there exists a norm preserving compact extension $\tilde{T}$: $Z \to X$.

The existence of norm preserving compact extensions for all the compact operators into a Banach space X implies that X^* is an L space in the sense of Kakutani [3] and X is polyhedral (cf. [11, theorem 7.10] where a stronger result is proved). Lindenstrauss raised the problem whether, conversely, a polyhedral Banach space whose dual is an L space has the "into" extension property for compact operators with no change of the norm [11, p. 102]. In the present paper we solve affirmatively this problem. In doing this, we are helped by a geometric characterization of the discussed spaces: there are no infinite dimensional w^*-closed proper faces in the closed unit balls of their duals. In [11, p. 103] a class of polyhedral Banach spaces which have the previously mentioned extension property was constructed. We show by an example that there are polyhedral Banach spaces whose duals are L-spaces and which do not belong to the class described in [11]. Section 2 contains a generalization of a linear extension theorem given in [10] which may be helpful outside the content of this paper.

1. We consider only Banach spaces over the real field. A Banach space is called polyhedral if the unit balls of all its finite dimensional subspaces are polytopes (cf. [7, p. 265]). The simplest example of such a space is c_0, the space of all real sequences converging to zero [7, p. 266]. A Banach space X is called a G space [11, p. 79] if X is isometric to a subspace of a $C(K)$ space, K compact Hausdorff consisting of all the functions satisfying a set Ω of relations of the form

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$$f(k_\alpha^1) = \lambda_\alpha f(k_\alpha^2), \quad k_\alpha^1, k_\alpha^2 \in K, \lambda_\alpha \text{ a scalar}, \alpha \in \Omega.$$

Any closed linear sublattice of $C(K)$ is a G space [4].

If X is a Banach space S_X denotes its closed unit ball. Let K be a compact convex subset of a linear topological space. By $\operatorname{ext} K$ we denote the set of extreme points of K. If K is symmetric about the origin we denote by $A_0(K)$ the Banach space of all continuous affine symmetric ($f(-k) = -f(k)$) real functions on K with the supremum norm. Any Banach space X can be identified with $A_0(S_{X^*})$, X^* being considered with its W^*-topology.

The convex hull of a set A is denoted by $\operatorname{co}(A)$. A convex subset of a convex set K is called a face of K if $\lambda k_1 + (1-\lambda)k_2 \in F$ with $k_1, k_2 \in K$ and $\lambda \in (0,1)$ imply $k_1, k_2 \in F$. If X is an L space and F is a face of S_X then the convex hull of F and $-F$ is called a facial section of S_X.

A map ψ from a convex subset K of a linear topological space to the collection of all the non-void convex subsets of a linear topological space E is called convex if

$$\lambda\psi(k_1) + (1-\lambda)\psi(k_2) \supseteq \psi(\lambda k_1 + (1-\lambda)k_2)$$

for all $k_1, k_2 \in K$ and $\lambda \in (0,1)$. It is lower semicontinuous if $\{k \in K: \psi(k) \cap U \neq \emptyset\}$ is open in the relative topology of K for each open subset U of E.

2. PROPOSITION 1. *Let X be a Banach space whose dual is an L space, H a w^*-closed facial section of S_{X^*} and Y a separable subspace of $A_0(H)$. There exists a linear isometry $T: Y \to X$ such that $x^*(T(y)) = y(x^*)$ for each $x^* \in H$, $y \in Y$.*

Proof. The idea of the proof is that of [1, theorem 5.2]. Let $\{y_n\}_{n=1}^\infty$ be dense in Y. The subspace of Y generated by $\{y_n\}_{n=1}^\infty$ induces a metrizable topology in Y^* which coincides with the w^*-topology on the w^*-compact set S_{Y^*}. Hence Y^* with this metrizable topology can be embedded into a Frechet space Z such that on S_{Y^*} the w^*-topology and the topology of Z are the same. Define the map $U: H \to S_{Y^*}$ by $U(x^*)(y) = y(x^*)$, $x^* \in H$, $y \in Y$. By [10, theorem 2.2,] U admits a w^*-continuous affine symmetric extension $U': S_{X^*} \to S_{Y^*}$. It is obvious that the map $T: Y \to A_0(S_{X^*}) = X$ given by $T(y)(x^*) = U'(x^*)(y)$, $x^* \in S_{X^*}$, $y \in Y$ has the needed properties.

REMARK. For $C(K)$ spaces, K compact Hausdorff, the above result was proved by Arens [1, theorem 5.2].

The next lemma is due essentially to Kalman [5]. Our statement is slightly stronger than that of [5] but has almost the same proof so we omit it.

LEMMA 2. *Let K be a polytope in a finite dimensional space, $\{v_i\}_{i=1}^n$ its vertices and $x_0 \in K$ with $x_0 = \sum_{i=1}^n \lambda_0^i v_i$, $\sum_{i=1}^n \lambda_0^i = 1$, $\lambda_0^i \geqq 0$. Then there are real continuous functions on K, $\{\lambda^i\}_{i=1}^n$, such that $x = \sum_{i=1}^n \lambda^i(x) v_i$, $\sum_{i=1}^n \lambda^i(x) = 1$, $\lambda^i(x) \geqq 0$ for each $x \in K$ and $\lambda^i(x_0) = \lambda_0^i$.*

3. We can now prove the main result of this paper:

THEOREM 3. *Let X be a Banach space whose dual is an L space. The following statements are equivalent.*

(1) *X is polyhedral.*

(2) *No subspace of X is isometric to c;*

(3) *S_{X^*} has no infinite dimensional w^*-closed proper faces;*

(4) *For every Banach spaces $Y \subset Z$ and every compact operator T: $Y \to X$ there exists a compact extension $\tilde{T}$: $Z \to X$ with $\| \tilde{T} \| = \| T \|$;*

(5) *For every Banach spaces $Y \subset Z$ and every operator T: $Y \to X$ with $\dim T(Y) \leqq 2$ there is a compact extension $\tilde{T}$: $Z \to X$ such that $\| \tilde{T} \| = \| T \|$.*

Proof. The implication (1) → (2) is obvious since c is not polyhedral.

(2) → (3). Assume that F is an infinite dimensional w^*-closed proper face of S_{X^*}. Then F is an infinite dimensional w^*-compact simplex, $H = \mathrm{co}(F \cup -F)$ is a w^*-closed facial section of S_{X^*} and $A_0(H)$ is isometrically isomorphic to the space of all continuous affine functions on F (normed by the supremum of the absolute value). It follows from [14] and [11, p. 67, corollary 2] that $A_0(H)$ contains an isometric copy of c. Proposition 1 implies then that X contains a subspace isometrically isomorphic to c.

(3) → (4). Assume (3) and let Y, Z and T be as in (4). We may suppose $\| T \| = 1$. Then the restriction of T^* to S_{X^*} maps S_{X^*} into S_{Y^*} and it is continuous when the first ball is considered with the w^*-topology and the second with the norm topology. We shall construct a continuous affine symmetric map χ from S_{X^*} (with the w^*-topology) into S_{Z^*} (with the norm topology) such that if $\phi: Z^* \to Y^*$ is the restriction map then $\phi(\chi(x^*)) = T^*(x^*)$ for each $x^* \in S_{X^*}$. Once this is done we are through since the operator $\tilde{T}: Z \to X$ defined by $\tilde{T}(z)(x^*) = \chi(x^*)(z)$ is an extension of T with $\| \tilde{T} \| = 1$ and its compactness follows from the fact that the restriction of $\tilde{T}^*$ to S_{X^*} is χ.

The map χ will be a selection of a certain set valued map from S_{X^*} into S_{Z^*}

but before we define it we are going to prove the following: there is a number $\alpha \in (0,1)$ such that only finitely many extreme points of $K = T^*(S_{X^*})$ are of norm greater than α. Assume that this claim is not true. Then there is a converging sequence $\{y_n^*\}_{n=1}^{\infty}$ in $\operatorname{ext} K$ with $y_n^* \neq y_m^*$ if $n \neq m$ whose limit y^* is of norm one. The intersection of K with a maximal convex subset of $\{y^* \in Y^*: \| y^* \| = 1\}$ is a closed face of K. The w^*-closed face $T^{*-1}(F) \cap S_{X^*}$ of S_{X^*} is finite dimensional by (3). Let $x_1^*, x_2^*, \cdots, x_p^*$ be its extreme points. Let x_{p+n}^* be an extreme point of S_{X^*} such that $T^*(x_{p+n}^*) = y_n^*$, $n = 1,2,\cdots$. We may suppose that $x_i^* \neq \pm x_j^*$ $i \neq j$. Now $\{x_i^*\}_{i=p+1}^{\infty} \cup (T^{*-1}(F) \cap S_{X^*})$ is a w^*-closed bounded subset of X^* so by the Milman theorem the extreme points of the w^*-closure of $\operatorname{co}(\{x_i^*\}_{i=1}^{\infty})$ are $\{x_i^*\}_{i=1}^{\infty}$. It follows from [13, proposition 1.2] that the w^*-closed convex hull of $\{x_i^*\}_{i=1}^{\infty}$ is $F' = \{ \sum_{i=1}^{\infty} a_i x_i^*: \sum_{i=1}^{\infty} a_i = 1,\ a_i \geqq 0\}$. Since X^* is an L space it is easy to see that F' is a (proper) face of S_{X^*}. Since F' is infinite dimensional we got a contradiction.

Let $\pm u_1^*, \pm_2^*, \cdots, \pm u_n^*$ be all the extreme points of K having norm one ($u_i^* \neq u_j^*$ if $i \neq j$). The set $K \cap \{y^* \in Y^*: \| y^* \| = 1\}$ is finite as a consequence of the above claim. Another consequence of this claim is the existence of a number $\beta \in (0,1)$ such that $y^* \in \operatorname{ext} K$ and $\| y^* \| > \beta$ imply $y^* = \pm u_i^*$ for a certain i and a suitable sign. Denote $K_\beta = K \cap \{y^* \in Y^*: \| y^* \| \leqq \beta\}$. By the Krein-Milman theorem and [2, p. 79] we have $K = \operatorname{co}(K_\beta \cup \{\pm u_i^*: 1 \leqq i \leqq n\})$ and if $y^* \in K$, $\| y^* \| = 1$ then $y^* \in \operatorname{co}\{\pm u_i^*: 1 \leqq i \leqq n\}$. Let $z_i \in S_{Z^*}$ be extensions of u_i^* to Z. Define a map ψ from K to the family of subsets of S_{Z^*} in the following way: if $y^* \in K$, $\| y^* \| < 1$ then

$$\psi(y^*) = \{z^* \in S_{Z^*}: \phi(z^*) = y^*\}$$

(recall that ϕ is the restriction map from Z^* to Y^*) and if $y^* \in K$, $\| y^* \| = 1$ then

$$\psi(y^*) = \left\{ \sum_{i=1}^{n} \lambda^i \varepsilon_i z_i : y^* = \sum_{i=1}^{n} \lambda^i \varepsilon_i u_i,\ \ \sum_{i=1}^{n} \lambda^i = 1,\ \lambda^i \geqq 0,\ \varepsilon_i = \pm 1 \right\}.$$

The set $\psi(y^*)$ is non-void closed convex and $\psi(-y^*) = -\psi(y^*)$ for any $y^* \in K$. For $\| y^* \| < 1$ these properties are obvious. For $\| y^* \| = 1$ they follow from the following observation: if the convex combinations $\sum_{i=1}^{n} \lambda^i \varepsilon_i u_i$ and $\sum \mu^i \varepsilon_i' u_i^*$ ($\varepsilon_i = \pm 1$, $\varepsilon_i' = \pm 1$) represent y^* and $\lambda^i \mu^i \neq 0$ for some i then $\varepsilon_i = \varepsilon_i'$ since there is a face of K containing y^* and included in the boundary of S_{Y^*}. The same observation helps us to see that $\psi(y^*)$ is convex.

We are going to prove that ψ is lower semicontinuous when K and S_{Z^*} are considered with their norm topologies. Let U be an open subset of Z^* and

$y_0^* \in \{y^* \in K: \psi(y^*) \cap U \neq \emptyset\} = M$. We have to show that y_0^* is an interior point of M (with the relative topology of K). If $\| y_0^* \| < 1$ there us $z_0^* \in \psi(y_0) \cap U$ with $\| z_0^* \| < 1$. Let $V = U \cap \{z^* \in Z^*: \| z^* \| < 1\}$. By the open mapping theorem $\phi(V)$ is open in Y^* hence $\phi(V) \cap K$ is a neighbourhood of y_0^* in K included in M. Suppose now that $\| y_0^* \| = 1$ and $z_0^* \in \psi(y_0^*) \cap U$ so $z_0 = \sum_{i=1}^n \lambda_0^i \varepsilon_0^i z_i^*$ with $y_0^* = \sum_{i=1}^n \lambda_0^i \varepsilon_0^i u_i^*$, $\sum_{i=1}^n \lambda_0^i = 1$, $\lambda_0^i \geqq 0$, $\varepsilon_0^i = \pm 1$, $1 \leqq i \leqq n$. Let B be an open ball of Z^* included in U of center z_0^* and radius $r > 0$. Let $\lambda^i(y^*)$, $\mu^i(y^*)$, $1 \leqq i \leqq n$, be non-negative continuous functions on the polytope $\mathrm{co}\{\pm u_i^*: 1 \leqq i \leqq n\}$ given by Lemma 2 such that

$$y^* = \sum_{i=1}^n \lambda^i(y^*)\varepsilon_0^i u_i^* - \sum_{i=1}^n \mu^i(y^*)\varepsilon_0^i u_i^*, \quad \sum_{i=1}^n \lambda^i(y^*) + \sum_{i=1}^n \mu^i(y^*) = 1$$

for each $y^* \in \mathrm{co}\{\pm u_i^*: 1 \leqq i \leqq n\}$ and $\lambda^i(y_0^*) = \lambda_0^i$, $u^i(y_0^*) = 0$, $1 \leqq i \leqq n$. Choose a neighborhood W_1 of y_0^* which satisfies for every $y^* \in W_1 \cap \mathrm{co}\{\pm u_i^*: 1 \leqq i \leqq n\}$:

$$\| \sum_{i=1}^n \lambda(y^*)\varepsilon_0^i z^* - \sum_{i=1}^n \mu^i(y^*)\varepsilon_0^i z_i^* - z_0^* \| < r/3.$$

An easy compactness argument shows that there exists a neighborhood W of y_0^* such that if $y^* \in W \cap K$ and $y^* = v y_1^* + (1-v) y_2^*$ with $y_1^* \in K_\beta$, $y_2^* \in \mathrm{co}\{\pm u_i^*: 1 \leqq i \leqq n\}$ and $v \in [0,1]$ then $v < r/3$ and $y_2^* \in W_1$. It is clear now that if $z^* \in \psi(y_1^*)$ the functional $vz^* + (1-v)[\sum_{i=1}^n \lambda^i(y_2^*)\varepsilon_0^i z_i^* - \sum_{i=1}^n \mu^i(y_2^*)\varepsilon_0^i z_i^*]$ belongs to $\psi(y^*) \cap B \subseteq \psi(y^*) \cap U$ so y_0^* is a relative interior point of M.

The map $\psi \circ T^*$ from S_{X^*} into the collection of non-void closed convex subsets of S_{Z^*} satisfies $\psi(T^*(x^*)) = -\psi(T^*(-x^*))$ for each $x^* \in S_{X^*}$ and it is convex and lower semicontinuous when S_{X^*} is taken with its relative w^*-topology and in S_{Z^*} we consider the norm topology. According to [10, theorem 2.2] it admits a continuous affine symmetric selection $\chi: S_{X^*} \to S_{Z^*}$. Obviously the map χ fulfills all the needed requirements.

The implication (4) → (5) is trivial. If X^* is an L space and satisfies (5) it follows from [11, theorem 7.9(b)] that each two dimensional subspace of X is polyhedral. By [6, theorem 4.7] X is polyhedral. This concludes the proof of the theorem.

Remarks. If a Banach space X has the property derived from (5) by changing the condition $\dim T(Y) \leqq 2$ into $\dim T(Y) \leqq 3$ then X^* is an L space (cf. [11, theorem 7.10]). It is not known whether the property (5) itself implies that X^* is an L space (see [11, p. 56] for a related conjecture). Assuming that X^* is an L

space it is not hard to give a direct proof of (3) → (1) based on Klee' s theorem mentioned above. If X^* is no longer assumed to be a L-space then, in general, the only valid implication between the properties (1), (2) and (3) is (1) → (2). Indeed, a smooth reflexive space of dimension greater than one satisfies (2) and (3) but it is not polyhedral. The closed unit ball of $l_1^* = l_\infty$ has infinite dimensional w^*-closed proper faces (e.g. the set $\{x = (x(n)) \in l_\infty : \|x\| = x(1) = 1\}$) even though l_1 contains no copy of c.

Corollary 1. *If X is a polyhedral Banach space and X^* is an L space then $X^* = l_1(\Gamma)$ for a suitable set Γ.*

Proof. Suppose first that X is separable. The set $\operatorname{ext} S_{X^*}$ is a metrizable G_δ [13, proposition 1.3] in the w^*-topology. If it is uncountable then by [8, p. 408, p. 445] it contains a sequence of distinct points which w^*-converges to an element of $\operatorname{ext} S_{X^*}$. It is easy to see then that S_{X^*} contains an infinite dimensional w^*-closed proper face and this contradicts (3) of the preceding theorem. Consequently S_{X^*} is countable and it follows from [13, proposition 1.2] that $X^* = l_1$. The general case follows now from [9, theorem 1.1].

The next corollary shows that the class of polyhedral spaces described in [11, p. 103] consists of all the polyhedral G spaces.

Corollary 2. *Let X be a G space. Then X is polyhedral if and only if X is isometric to a subspace X' of a $C(K)$ space, K compact Hausdorff, consisting of all the functions which satisfy a set Ω of relations*

$$f(k_\alpha^1) = \lambda_\alpha f(k_\alpha^2), \quad k_\alpha^1, k_\alpha^2 \in K, \ \lambda_\alpha \in (-\infty, \infty), \ \alpha \in \Omega$$

with the following property: for each $f \in X'$, $f \neq 0$, there is a finite set $\{k_i\}_{i=1}^n \subseteq K$ such that

$$\|f\| > \sup\{|f(k)| : k \in K, \ k \neq k_i\}.$$

Proof. Suppose that X is a polyhedral G space and let K be the w^*-closure of $\operatorname{ext} S_{X^*}$. Then by [12, theorems 2 and 2′] the canonical image of X in $C(K)$ is a subspace consisting of all the functions which satisfy a set of conditions of the above form. Assume now that there is $x \in X$, $x \neq 0$, such that

$$\|x\| = \sup\{|x(k)| : k \in K - C\}.$$

whenever C is a finite subset of K. It follows that there is an infinite sequence $\{k_n\}_{n=1}^\infty$ of distinct points of K such that $|x(k_n)| \to \|x\|$. Without loss of generality we may assume that $x(k_n) \to \|x\|$, $\{k_n\}_{n=1} \subseteq \operatorname{ext} S_{X^*}$ and $k_n \neq \pm k_m$ if

$n \neq m$. The set $F = \{x^* \in S_{X^*}: x^*(x) = \|x\|\}$ is a proper w^*-closed face of S_{X^*} which must be finite dimensional by (3) of Theorem 3. Clearly $F \cup \{k_n\}_{n=1}$ is w^*-closed. As in the proof of Theorem 3 one may show that the w^*-closed convex hull of $F \cup \{k_n\}_{n=1}^{\infty}$ is an infinite dimensional face of S_{X^*} and this is a contradiction. Hence for each $x \in X$, $x \neq 0$, there is a finite set $C_x \subseteq K$ such that

$$\|x\| > \sup\{|x(k)|: k \in K - C_x\}.$$

The other half of the corollary was proved in [11, p. 103–104].

We give now an example of a polyhedral Banach space X which is not a G space but $X^* = l_1$. This example settles a problem raised in [12]. Let X be the space of all the convergent real sequences $x = \{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = \frac{1}{2}x_1 + \frac{1}{3}x_2$. It is easy to check that the vectors

$$e_1 = (1, 0, \tfrac{1}{2}, \tfrac{1}{2}, \cdots),$$

$$e_2 = (0, 1, \tfrac{1}{3}, \tfrac{1}{3}, \cdots),$$

$$e_n = (0, 0, 0, \cdots, 0, 1, 0, \cdots), \qquad n \geqq 3$$

(1 in the nth place) form a basis of X. The subspace X_n spanned by $\{e_i\}_{i=1}^{n}$ is isometrically isomorphic to l_∞^n. Since $X_n \subseteq X_{n+1}$ and $\bigcup_{n=1}^{\infty} X_n$ is dense in X it follows from [11, p. 66, corollary 1] that X^* is an L space. Actually $X^* = l_1$ since X is a hyperplane of c. The extreme points of S_{X^*} are the functionals $\pm x_n^*$ where $x_n^*(x) = x_n$. The only w^*-limit points of these functionas are $\pm(\frac{1}{2}x_1^* + \frac{1}{3}x_2^*)$, both of norm less than one so S_{X^*} has no infinite dimensional w^*-closed proper faces. By Theorem 3 X is a polyhedral space. However, X is not a G space. This results for instance from the fact that there is no element $x \in X$ such that $x^*(x) = \max(x^*(e_1), x^*(e_2), 0) + \min(x^*(e_1), x^*(e_2), 0)$ for all $x^* \in \operatorname{ext} S_{X^*}$ (see [12, theorem 2]).

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